Optimal Product Design: A CAPM approach

Marc Oliver Rieger*, Mei Wang† and Thorsten Hens‡

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Abstract

We study properties of structured financial products optimizing a utility functional of a customer. The conventional method may have the disadvantage that the *a priori* restriction to a certain number of assets could make it impossible to find the optimal portfolio. So instead of optimizing the distribution of given assets, we impose only the price constraint as given by the CAPM and optimize the return distribution. In particular on nowadays markets where a multitude of asset types is available, it seems helpful to optimize first in the general framework, assuming a complete market, and then to find assets whose return distribution and conjoint probability distribution with the market portfolio resemble the theoretically optimal portfolio as closely as possible. We introduce a method to construct such optimal portfolios numerically and present some results for the cases of expected utility and cumulative prospect theory.

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1 Introduction

1.1 CAPM Theory

The mean-variance analysis goes back to Harry Markowitz [4]. In his work "Portfolio Selection" he recommended the use of an expected return-variance of return rule, "...both as a hypothesis to explain well-established investment behavior and as a maxim to guide one's own action". Later, Jagannathan and Wang [3] recognized the mean-variance analysis and the Capital Asset Pricing Model (CAPM) as "...the major contributions of academic research in the post-war era." Campbell and Viciera [1] wrote: "Most MBA courses, for example,

*ETH Zürich, Department of Mathematics, marc.rieger@math.ethz.ch
†University of Zürich, Swiss Banking Institute, wang@isb.unizh.ch
‡University of Zürich, Swiss Banking Institute, thens@iew.unizh.ch

still teach mean-variance analysis as if it were a universally accepted framework for portfolio choice".

This celebrated theory allows to compute the price of a given asset, once its return distribution and its covariance with the market portfolio are known. Let the returns of the asset be R_j and the returns of the market be R_M and let their mean and variance be given by μ_j, μ_M and σ_j, σ_M, respectively. Let μ_f be the mean value of the risk-free asset (i.e. the interest rate). Then we have:

$$\mu_j - \mu_f = \beta_{jM}(\mu_M - \mu_f) \text{ where } \beta_{jM} = \frac{\operatorname{cov}(R_j, R_M)}{\sigma_M^2}. \tag{1}$$

The underlying assumption of an efficient market in which all market participants act according to a maximization of a utility depending only on the mean and the variance of an asset, is relatively strong, but nevertheless widely used. In this article, we take this model as a description of the financial market. We then aim to find an optimal portfolio for an investor on this market who is *not* following this mean-variance maximization, but who is either rational (and hence follows Expected Utility Theory) or who is following the most commonly used model for actual behavior under risk, the Cumulative Prospect Theory which we will introduce in the next section.

1.2 Expected utility theory and CPT

There are mainly two approaches to an understanding of Expected Utility Theory (EUT): the first one was introduced by Bernoulli [**?**] who argued that the value of a lottery (where several payoffs are possible and their probabilities are known) cannot be naively computed by the expected value of the payoff, but has to take into account the different *utility* that the payoffs would give to a person. Hence one should take the expected value *of the utility of the payoff*. Mathematically, this leads to the formula

$$EUT(A) = \mathbb{E}(u) = \sum_i u(x_i)p_i$$

for the utility of a lottery A with outcomes x_i and associated probabilities p_i. Here $u(x)$ is the utility of the outcome x.

The second approach to this theory has been found by von Neumann and Morgenstern [**?**]. They stated three very natural axioms for rational decisions between lotteries (transitivity, continuity and independence of irrelevant alternatives) and then proved that these axioms already guarantee the existence of a utility function in the spirit of Bernoulli.

This property makes EUT the "gold standard" of rational decisions in situations which involve risk.

It turned out, however, that people do not always decide in a way that is compatible with EUT. There are in fact some systematic deviations, and this led to the development of new *descriptive* theories. The most prominent example for such theories is the Cumulative Prospect Theory (CPT) by Tversky and Kahneman [6].

The key ideas of [6] were the following:

1. Instead of evaluating the EUT in terms of the total wealth (which would be rational), only changes to the current level of wealth (gains and losses) are taken into consideration. The corresponding utility is concave (i.e. risk-averse) in gains, but convex (risk-seeking) in losses.

2. They replaced the probabilities of EUT by *differences of cumulative probabilities.* In other words, we replace p_i with the expression $w(F_i)-w(F_{i-1})$, where w is a certain nonlinear function and $F_i := \sum_{j=1}^{i} p_j$ are cumulative probabilities. (We set $F_0 := 0$.) Of course, the order of the events is now important, and we order them in the natural way, i.e. by the amount of their outcomes. If we choose w appropriate, this leads to an underweighting of very low and very high outcomes which is frequently observed.

With these two main changes, several puzzles of behavior can be resolved, for instance that people tend to buy insurances, but take part in lotteries. (In both cases, they overweight extreme outcomes like being robbed or winning the lottery.)

The prototypical example for a utility function for CPT has been given in [6] for $\alpha \in (0,1)$ and $\lambda > 1$:

$$u(x) := \begin{cases} x^\alpha, & x \geq 0 \\ -\lambda(-x)^\alpha, & x < 0. \end{cases} \tag{2}$$

The average values of α is approx. 0.8. The so-called "loss-aversion" λ is usually set to approx. 2.

The probability weighting function w was originally given as

$$w(F) := \frac{F^\gamma}{(F^\gamma + (1-F)^\gamma)^{1/\gamma}}. \tag{3}$$

The value of γ is approximately 0.6.

The complete formula of Cumulative Prospect Theory is then as follows:

Definition 1.1 (Cumulative Prospect Theory) *For a lottery A with n outcomes $x_1, \ldots, x_n$ and probabilities $p_1, \ldots, p_n$ where $x_1 < x_2 < \ldots < x_n$ and $\sum_{i=1}^{n} p_i = 1$ we define*

$$CPT(A) := \sum_{i=1}^{n} \left(w(F_i) - w(F_{i-1})\right) v(x_i), \tag{4}$$

where $F_0 := 0$ and $F_i := \sum_{j=1}^{i} p_j$ for $i = 1, \ldots, n$.

There exist slightly different definitions of the CPT functional. In particular the original formulation in [6] differed in that it used the above formula only for losses, but a *de*-cumulative probability (i.e. $F_i := \sum_{j=i+1}^{n} p_j$) for gains. In finance, however, the above formula is more frequently used, since it is structurally simpler and essentially equivalent with the original formulation if one allows for changes in the weighting function.

2 Finding optimal portfolios

An optimal portfolio is an asset allocation that optimizes a person's utility. This utility may either be measured in the mean-variance sense, and then the famous Two-Fund Separation Theorem tells us that an optimal portfolio on a CAPM market is always a combination of the risk free asset and a market portfolio. Or it could be measured by the Expected Utility Theory or Cumulative Prospect Theory or in other ways.

In this article we deal with the situation of EUT- and CPT-optimization. These are in a certain sense the most important variants, since EUT is the model of rational decisions and CPT is the most widely accepted descriptive model for decisions under risk.

In the following, we first present the standard method for an optimization, before we present our approach.

2.1 Optimal allocations

The standard method for portfolio optimization starts from a selection of assets and computes the optimal distribution of these assets in a portfolio. The underlying functional that is maximized can be, e.g., Expected Utility or CPT. The optimal portfolio is fully described by the relative weights of the selected assets. The advantage is clear, since this method can be immediately applied: we just need to buy the computed amount of each asset from the list to construct the optimal portfolio.

The drawback of this method is the inherent restriction posed by the *a priori* selection of assets. Depending on the kind of selection, very different portfolios can be the result. As an extreme example imagine a bold investor with a long time horizon who does not refrain from taking risks. If the assets on which his portfolio is optimized are all low-risk bonds, his "optimal" portfolio built up by those assets will yield a low average return and will hence not be optimal for him at all. Now it is easy to adjust the overall "riskiness" of a portfolio by adding or subtracting risky assets, however, if we want to take into account more subtle differences in the preference structure of investors, it is not at all clear how an optimal portfolio should look like and what type of assets it should contain – given the immense variety of financial products on the market.

2.2 A general optimization strategy

When selecting a number of assets and finding their optimal combination, we inevitably restrict ourselves by the selection of the assets. To avoid this, we try a different starting point and assume a complete market. We optimize not the allocation of given assets, but instead the return distribution of the total portfolio. In other words, we optimize over all probability measures on the space of possible returns. Without a certain constraint, this will obviously not work, since the optimal portfolio will always be the one that gives the largest possible return for sure. What we have to take into account is the availability of an asset on the market. In a complete market, every possible return distribution is available, albeit for a certain price, which has to be subtracted from its gross return. If we have a certain wealth we can only pay a certain price, and hence we can only search for the the optimal outcome distribution among all outcome distributions up to this price. The price of an outcome distribution, however, is given by the CAPM formula. Hence our optimization problem becomes the following:

Maximize the utility of p among all probability measures p that satisfy

$$\mathbb{E}(p) - \mu_f \quad = \quad \frac{\operatorname{cov}(p, M)}{\sigma_M^2}(\mu_M - \mu_f).$$

We can simplify this formula substantially when we assume that the utility of the investment will only depend on the return distribution, but not, e.g., on its relation to the market portfolio. This can be justified when we consider that we mostly aim to study optimal portfolios for *private investors*. – Whereas a bank might want to invest in a way that mitigates certain risks that it is facing by giving loans etc., a private investor does not take such problems into account, in particular when we assume that we discuss how to invest the *whole wealth* of an investor. We are hence excluding the possibility that an investor wants to invest some part of his wealth in a way to ensure himself from the risk that other parts of his investments are facing.

If we assume that an investor follows EUT or CPT maximization in the usual sense, we implicitly made already the assumption that the utility only depends on p, but not on M or other factors.

A result from [2] shows now that the conjoint probability T between p and the market portfolio M must maximize the covariance of p and M. One sometimes refers to such a maximal conjoint probability as "monotone". It is now possible to compute the covariance between p and M easily, using an algorithm from [5]. We will discuss this in more details in the next section.

3 Numerical optimization

In this section we present a numerical method to compute optimal outcome distribution in the sense defined in the previous section. We consider a model with only finitely many possible outcomes. Although the general case can be approximated by such lotteries, there are some difficulties to overcome when allowing for arbitrary lotteries, since we can in general not expect that the optimization problem will have a solution. The reason for this lies in the inherent difficulties of the Mean-Variance Approach which are illustrated by the famous mean-variance paradox. The CAPM pricing formula gives too low prices for strongly skewed distributions, since the (symmetric) variance does not distinguish between a risk of losing or a risk of winning money. Therefore the price constraint of the CAPM formula is not always sufficient to exclude arbitrarily good assets when measured with EUT or CPT. The circumvention of this problem will be an interesting task for further studies. Here we restrict ourselves to the finite case, where the existence is trivially satisfied.

3.1 Underlying algorithm

Let $x_1, \ldots, x_n$ be the set of possible outcomes, where $x_1 < x_2 < \ldots < x_n$. We want to find the optimal vector $(p_1, \ldots, p_n)$, where $p_i \geq 0$, of probabilities for these outcomes such that:

(i) The total probability is one: $p_1 + \ldots + p_n = 1$.

(ii) The asset, described by the outcomes x_i and their probabilities p_i has a price of (at most) π, where π is given. This corresponds to the constraint (1).

Looking at the constraint, it seems that we have not only to optimize over the vector p, but also over its conjoint probability with M, in order to compute the covariance. This would lead to an optimization problem in n^2 variables, rather than in n variables, which would make the numerical computation much harder.

Fortunately, we know from the theoretical considerations of the last section that an optimal portfolio will correspond to the *maximal* covariance, given p and M. Hence we do not have to optimize over the conjoint probability distribution, but can instead compute the maximal covariance of the measures p and M, where M is the outcome distribution of the market and $p = (p_1, \ldots, p_n)$. The following algorithm, taken from [5], does this computation:

First, compute the mean values μ_p and μ_M.
Set $i = j = 1$, $L = M_1$, C=0.
While $i \leq n$ or $j \leq n$:
{
If $L > p_j$ then $L = L - p_j$, $C = C + p_j(x_i - \mu_p)(x_j - \mu_M)$.
If $L \leq p_j$ then $L = 0$, $C = C + L(x_i - \mu_p)(x_j - \mu_M)$.
If $L = 0$ then $i = i + 1$, $L = M_i$, otherwise $j = j + 1$.
}

The algorithm terminates since $\sum_{i=1}^{n} M_i = 1 = \sum_{j=1}^{n} p_j$. The variable C gives the maximal covariance of p and M.

Using this algorithm, the constraint (1) can be computed without explicitly knowing the conjoint probability between p and M. The resulting finite constrained maximization problem can be solved with standard algorithms for non-concave maximization.

This can potentially lead to problems, since these algorithms do not always find the global minimizer. In the case of relatively few potential outcomes, however, the dimension of the problem is still low enough to ensure convergence in most optimizations.

3.2 Numerical results

We assume eight different possible outcomes for an investment: a loss of 5%, a loss of 3% and so on up to a gain of 9%. As risk-free rate we choose a return of

1%, and the market portfolio follows a normal distribution (or more precisely, its approximation by the finitely many outcomes) with a mean return of 4% and a variance of 4%.

We first consider an expected utility investor whose utility for a return of x% is described by $(x + 10)^{\alpha}$ with $\alpha = 0.88$. The resulting degree of risk aversion is not sufficient to counter-balance the attractive mean of a high-risk investment as it can be found in the CAPM pricing formula: the optimal portfolio loses 5% in a quarter of the cases, but wins 9% in nearly three quarter of the cases (see Table 1).

Table 1: Optimal outcome distributions in a CAPM market for an EUT investor.

Example 1		Example 2	
Return(%)	Probability(%)	Return(%)	Probability(%)
-5	25.0	-6	25.0
-3	0.0	-4	0.0
-1	0.4	-2	0.0
1	0.5	0	0.0
3	0.2	2	0.0
5	0.0	4	0.0
7	0.1	6	50.0
9	73.8	8	25.0

The above representation is quite unusual. Financial products are more frequently described by a diagram of the market return vs. the portfolio return. We can convert our portfolio into such a diagram, since we know that its returns are a monotone function of the market return (compare [2]). The dependence of the portfolio return from the market return in the case of EUT is sketched in Fig. 1.

The results are quite stable under small changes of the set of possible returns: if we consider the possible outcomes $-6\%, -4\%, \ldots, 8\%$ we end up with a qualitatively similar distribution (Table 1). The remaining difference can be explained with the problems caused by the nonconvex optimization.

A problem is that the "extreme" outcomes play an important role, and that a model with different "extreme" outcomes may lead to a very different optimal return distribution. This difficulty is caused by the underlying assumption of a market described by the Capital Asset Prizing Model and is strongly related to the Mean-Variance Paradox.

Let us turn our attention now to the optimization for a CPT client. The first example in Table 2 shows the case of a CPT client with the parameter values measured by [6] ($\alpha = 0.88$, $\lambda = 2.25$, $\gamma = 0.67$). The optimal portfolio has a very low chance to yield less than -1%. Its returns are with high probability close to the fixed interest rate. Additionally there is a relatively small chance for the highest available return. If we translate this into a diagram of the market

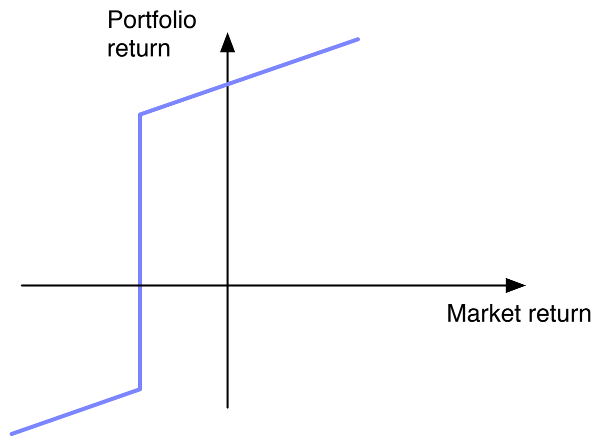

Figure 1: Dependence of the portfolio return on the market return of an optimal portfolio in the case of EUT (schematic picture).

return vs. the portfolio return (Fig. 2), we notice that at the lower end of the returns, the losses are bounded, and that on the upper end, there is a small possibility for a large return. This corresponds to a capital protection against medium to large losses, plus a participation rate which increases strongly for large returns.

When we change the risk coefficient α from 0.88 into 1 (Example 2 in Table 2), the optimal distribution is similar to the first example, but with a slightly higher chance to obtain a return rate of 9%. The third example deals with a CPT client with linear weighting function, and the fourth example shows the optimal portfolio of a CPT client without loss aversion, i.e., λ is equal to 1 instead of 2.25. In both cases, the client prefers a portfolio with larger variance, and is more tolerant to potential losses which results in a pattern very similar to the one of the EUT client of Table 1.

In general, both CPT and EUT investors seem to prefer bimodal distributions, probably because this maximizes the covariance. Our numerical results suggest that a typical CPT investor would prefer his return distribution to be above a certain threshold. Loss aversion ($\lambda > 1$) and a probability over-/underweighting ($\gamma < 1$, which makes a small chance for a large gain particularly tempting) are both crucial for the effect, as the Examples 3 and 4 demonstrate. The "capital protection" is already frequently included into the design of financial products. Many financial institutions provide capital protection to their clients so that the investors are guaranteed to get at least part of their original capital at the end of maturity. The high-return/small-probability part of the optimal distribution, however, is rarely included into financial products. A notable exception is the so-called "Gewinnsparen" in Germany, a traditional investment form, where a

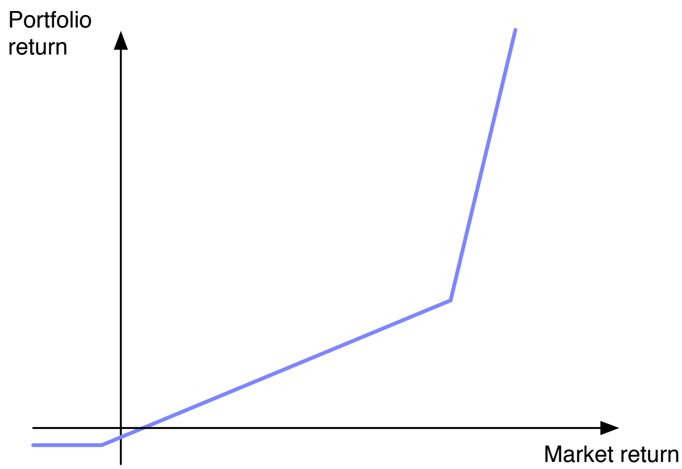

Figure 2: Dependence of the portfolio return on the market return of an optimal portfolio in the case of CPT (schematic picture).

Table 2: Optimal outcome distributions in a CAPM market for an CPT investor.

	Example 1	Example 2	Example 3	Example 4
α	0.88	1	0.88	0.88
γ	0.67	0.67	1	0.67
λ	2.25	2.25	2.25	1
Return(%)	Probability(%)	Probability(%)	Probability(%)	Probability(%)
-5	0.9	0.0	25.0	25.0
-3	0.0	0.0	0.0	0.0
-1	7.0	25.0	0.0	0.4
1	81.3	60.8	0.0	0.5
3	1.8	0.0	0.0	0.2
5	0.9	0.0	8.0	0.0
7	0.6	0.0	9.6	0.1
9	7.5	14.2	57.4	73.8

conservative fixed-interest bank account is combined with a lottery for which a very small amount of the invested money is used. This leads to a small chance of winning a relatively large prize without any risk to lose money, which mimics our theoretically optimal return distribution. However, although our results imply that investment products with capital protection plus a “lottery” can be appealing to typical CPT agents, the real-life attractiveness of such investments may of course also be influenced by other factors than their payoff distributions.

4 Conclusions

We have used a new approach to the design of financial products that is solely based on a pricing formula and not on a selection of assets and its combination. Our approach allows to discover general properties of optimal financial products that can then be constructed by either combining existing assets or introducing them as new innovations on the market. We have studied a simple instance of this optimization procedure by considering a model market with only finitely many possible outcomes. The generalization to arbitrarily many outcomes turns out to be difficult due to technical limitations of the Capital Asset Pricing Model that we have used as starting point for our consideration. This problem is closely related to the Mean-Variance Paradox. To overcome this limitation, we will need to consider more general market models, instead of CAPM. Nevertheless, already in this simple case, we were able to discover interesting properties of optimal financial products. In particular, we found that capital protection plays an important role in behavioral finance, and we could demonstrate that it is caused by a combination of probability overweighting and loss aversion. Moreover, we found that a small “lottery component” (corresponding to an increased participation rate for large returns) could be an appealing innovation for CPT investors.

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References

[1] John Y. Campbell and Luis Viceira. *Strategic Asset Allocation: Portfolio Choice for Long-Term Investors.* Oxford University Press, 2002.

[2] Thorsten Hens and Marc Oliver Rieger. Following the market. in preparation.

[3] Ravi Jagannathan and Zhenyu Wang. The conditional capm and the cross-section of expected returns. *Journal of Finance*, 51(1):3–53, 1996.

[4] Harry M. Markowitz. Portfolio selection. *The Journal of Finance*, 7(1):77–91, March 1952.

[5] Marc Oliver Rieger. Monotonicity of transport plans and applications. submitted.

[6] Amos Tversky and Daniel Kahneman. Advances in Prospect Theory: Cumulative representation of uncertainty. *Journal of Risk and Uncertainty*, 5:297–323, 1992.